Planetary Physics (10 points)

Part A. Mid-ocean ridge (5.0 points)

A.1 (0.8 points)

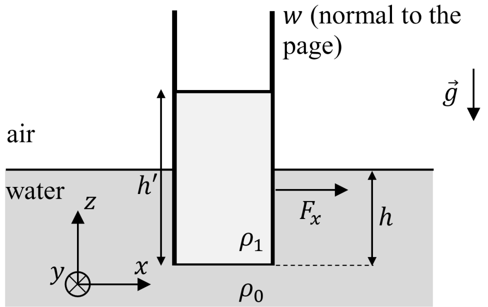

Figure 1

Let h' be the height of the column of oil (see Fig. 1). Then pressure at depth h below the water surface must be $p_h = \rho_0 g h = \rho_{\text{oil}} g h'$, from where $h' = \frac{\rho_0}{\rho_{\text{oil}}} h$. Horizontal force on the plate $F_x = F_1 - F_0$, where the force due to new fluid is $F_1 = \frac{\rho_{\text{oil}} g h'}{2} \cdot h' w$ and the force due to water is $F_0 = \frac{\rho_0 g h}{2} \cdot h w$.

Combining all the equation above, we get

$$F_x = \left(\frac{\rho_0}{\rho_{\text{oil}}} - 1\right) \frac{\rho_0 g h^2 w}{2}.$$

This force acts on the right plate to the right.

A.2 (0.6 points)

Consider a rectangular mass element of the crust. Since relation $l(T) = l_1 \left[1 - k_l \left(T_1 - T\right) / \left(T_1 - T_0\right)\right]$ holds for all three dimensions of the solid, its volume V satisfies

$$V = V_1 \left(1 - k_l \frac{T_1 - T}{T_1 - T_0}\right)^3,$$

where V_1 is the volume at $T = T_1$. If the mass of the element is m, density is then

$$\rho(T) = \frac{m}{V} = \frac{m}{V_1} \left(1 - k_l \frac{T_1 - T}{T_1 - T_0}\right)^{-3} = \rho_1 \left(1 - k_l \frac{T_1 - T}{T_1 - T_0}\right)^{-3}.$$

Since $k_l \ll 1$, this can be approximated as

$$\rho\left(T\right) \approx \rho_1 \left(1 + 3k_l \frac{T_1 - T}{T_1 - T_0}\right),$$

so that $k = 3k_l$.

A.3 (1.1 points)

Since mantle behaves like a fluid in hydrostatic equilibrium, pressure $p(x, z)$ at $z = h + D$ must be the same for all x. Therefore,

$$p(0, h + D) = p(\infty, h + D).$$

Similarly, we must have

$$p(0, 0) = p(\infty, 0).$$

Hence, the change in pressure between $z = 0$ and $z = \infty$ must be the same at both $x = 0$ and $x = \infty$. At the ridge axis

$$p(0, h + D) - p(0, 0) = \rho_1 g (h + D),$$

while far away

$$p(\infty, h + D) - p(\infty, 0) = \rho_0 g h + \int_h^{h+D} \rho(T(\infty, z))\, g\, \mathrm{d}z.$$

Since the temperature of the crust at $x = \infty$ depends linearly on height, after applying the relevant temperature boundary conditions,

$$T(\infty, z) = T_0 + (T_1 - T_0)\frac{z - h}{D}.$$

From all the equations above and by using the density formula given in the problem text,

$$\rho_1 g (h + D) = \rho_0 g h + \int_h^{h+D} \rho_1 \left(1 + k\frac{T_1 - T_0 - (T_1 - T_0)\frac{z-h}{D}}{T_1 - T_0}\right) g\, \mathrm{d}z,$$

from where we straightforwardly obtain

$$D = \frac{2}{k}\left(1 - \frac{\rho_0}{\rho_1}\right) h.$$

A.4 (1.6 points)

The net horizontal force on the half of the ridge is the difference between the pressure forces acting at $x = 0$ and $x = \infty$:

$$F = L\int_0^{h+D} p(0, z)\, \mathrm{d}z - L\int_0^h p(\infty, z).$$

From considerations of the previous question, pressure at $x = 0$ is

$$p(0, z) = p(0, 0) + \rho_1 g z,$$

while very far away

$$p(\infty, z) = p(\infty, 0) + \rho_0 g z$$

The equations above can be combined into

$$F = L\int_0^{h+D} \left(p(0, 0) + \rho_1 g z\right) \mathrm{d}z - L\int_0^h \left(p(\infty, 0) + \rho_0 g z\right) \mathrm{d}z.$$

After a straightforward integration and using $p(0,0) = p(\infty, 0)$,

$$F = Lp(0,0)\,D + L\rho_1 g\frac{(h+D)^2}{2} - L\rho_0 g\frac{h^2}{2}.$$

Since $k \ll 1$, and $D \propto k^{-1}$, the term with $D^2 \propto k^{-2}$ is of the leading order, hence, after substituting the result of A.3, the required answer is

$$F \approx \frac{2gLh^2(\rho_1 - \rho_0)^2}{k^2\rho_1}.$$

A.5 (0.9 points)

Method 1: dimensional analysis. The timescale τ is expected to depend only on density of the crust ρ_1, its specific heat c, thermal conductivity κ and thickness D. Hence, we can write

$$\tau = A\rho_1^\alpha c^\beta \kappa^\gamma D^\delta,$$

where A is a dimensionless constant. We will obtain the powers α–δ via dimensional analysis.

Define the symbols for different dimensions: L for length, M for mass, T for time and Θ for temperature. Then τ, ρ_1, c, κ and D have dimensions T, $\mathrm{ML^{-3}}$, $\mathrm{L^2T^{-2}\Theta^{-1}}$, $\mathrm{MLT^{-3}\Theta^{-1}}$ and L, respectively. The resulting set of linear equations to balance the powers of length, mass, time and temperature, respectively, is

$$\begin{cases} 0 = -3\alpha + 2\beta + \gamma + \delta, \\ 0 = \alpha + \gamma, \\ 1 = -2\beta - 3\gamma, \\ 0 = -\beta - \gamma. \end{cases}$$

This gives $\alpha = \beta = 1$, $\gamma = -1$, $\delta = 2$. Hence,

$$\tau = A\frac{c\rho_1 D^2}{\kappa}.$$

Method 2: scale analysis. Consider a piece of crust of area S. Heat flux that has to be transmitted through the crust is of order $Q \sim c\rho_1 SD\Delta T$, where $\Delta T = T_1 - T_0$. On the other hand, the law of thermal conductivity gives that $\kappa\frac{\Delta T}{D} \sim \frac{Q}{S\tau}$.

From the two equations, $c\rho_1 SD\Delta T \sim S\tau\kappa\frac{\Delta T}{D}$, from where we get that τ is independent of ΔT and

$$\tau \sim \frac{c\rho_1 D^2}{\kappa}.$$

Part B. Seismic waves in a stratified medium (5.0 points)

B.1 (1.5 points)

Seismic waves in this problem can be treated by using ray theory. Namely, their propagation is described by the Snell's law of refraction

$$n(0)\sin\theta_0 = n(z)\sin\theta,$$

where the refractive index is

$$n(z) = \frac{c}{v(z)} = \frac{c}{v_0\left(1+\frac{z}{z_0}\right)}$$

and c denotes the seismic wave speed in a material with refractive index $n = 1$. From the two equations above we have

$$v_0\left(1+\frac{z}{z_0}\right)\sin\theta_0 = v_0\sin\theta.$$

Method 1. Since this describes an arc of a circle, we have that at $\theta = \frac{\pi}{2}$, $z = R - R\sin\theta_0$ (Fig. 2), giving

$$\left(1+\frac{R-R\sin\theta_0}{z_0}\right)\sin\theta_0 = 1,$$

from where the circle radius $R = \frac{z_0}{\sin\theta_0}$. From simple geometry we get

$$x_1(\theta_0) = 2R\cos\theta_0,$$

leading to

$$x_1(\theta_0) = 2z_0\cot\theta_0,$$

i.e. $A = 2z_0$ and $b = 1$.

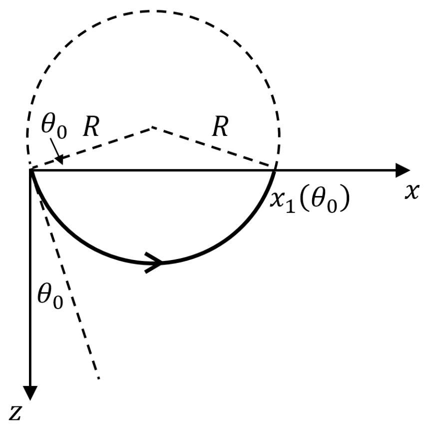

Figure 2

Method 2. Implicitly differentiating $v_0\left(1+\frac{z}{z_0}\right)\sin\theta_0 = v_0\sin\theta$ gives

$$\frac{\mathrm{d}z}{z_0}\sin\theta_0 = \cos\theta\,\mathrm{d}\theta.$$

An infinitesimal ray path length $\mathrm{d}l$ is related to the change in the vertical coordinate via

$$\mathrm{d}z = \mathrm{d}l\,\cos\theta,$$

giving

$$\mathrm{d}l = \frac{z_0}{\sin\theta_0}\,\mathrm{d}\theta.$$

This is an equation of an arc of a circle of radius $R = \frac{z_0}{\sin\theta_0}$

Alternatively, instead of considering an infinitesimal ray path length $\mathrm{d}l$, one can obtain the answer by writing

$$\cot\theta = \frac{\mathrm{d}z}{\mathrm{d}x} = \frac{\mathrm{d}z}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\mathrm{d}x}.$$

The first derivative can be eliminated via Snell's law, leading to

$$\cot\theta = \frac{z_0\cos\theta}{\sin\theta_0}\frac{\mathrm{d}\theta}{\mathrm{d}x},$$

which can be integrated to get

$$x_1 = -\frac{z_0}{\sin\theta_0}\int_{\text{start}}^{\text{end}}\mathrm{d}\cos\theta = \frac{2z_0\cos\theta_0}{\sin\theta_0},$$

where we used Snell's law again to get that the ray has $\cos\theta = -\cos\theta_0$ at the point where it reaches the surface.

B.2 (1.5 points)

In two dimensions, $\frac{E}{\pi}\,\mathrm{d}\theta_0$ is the energy carried by rays that are emitted within interval $[\theta_0, \theta_0+\mathrm{d}\theta_0)$. On the other hand, the energy carried by rays that arrive at $[x, x+\mathrm{d}x)$ is $\varepsilon\,\mathrm{d}x$. Therefore,

$$\varepsilon = \frac{E}{\pi}\left|\frac{\mathrm{d}\theta_0}{\mathrm{d}x}\right|.$$

Using the result of question B.1,

$$\frac{\mathrm{d}x}{\mathrm{d}\theta_0} = -\frac{Ab}{\sin^2(b\theta_0)} = -Ab\left(1+\cot^2(b\theta_0)\right) = -\frac{b\left(A^2+x^2\right)}{A}.$$

Hence,

$$\varepsilon(x) = \frac{EA}{\pi b\left(A^2+x^2\right)} = \frac{2Ez_0}{\pi\left(4z_0^2+x^2\right)}.$$

This function is plotted in Fig. 3.

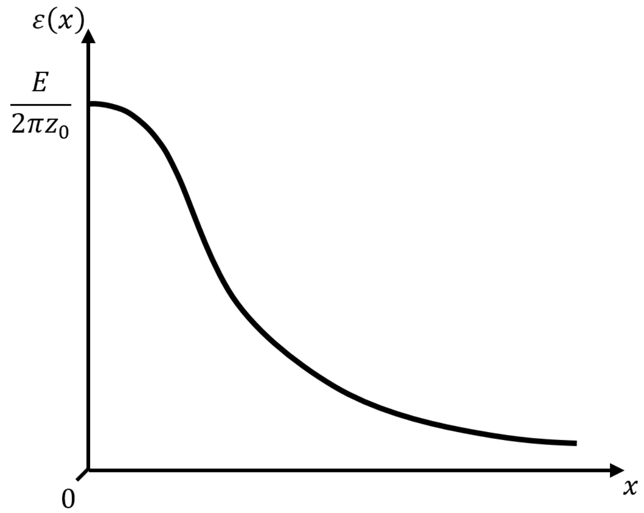

Figure 3. Plot of the function $\varepsilon(x)$.

B.3 (2.0 points)

Define $x_- = x_1\left(\theta_0 - \frac{\delta\theta_0}{2}\right)$ and $x_+ = x_1\left(\theta_0 + \frac{\delta\theta_0}{2}\right)$. To the leading order in $\delta\theta_0$, $x_- \approx x_+ \approx x_1(\theta_0)$. With each reflection of the signal, the horizontal distance between the points where the edges of the signal reflect increases by $|x_+ - x_-| = x_- - x_+$. When moving along the positive x-axis, these zones get wider until they overlap. If this happens after N reflections, then

$$N \approx \frac{x_1(\theta_0)}{x_- - x_+},$$

where the approximate sign tends to equality as $\delta\theta_0 \to 0$.

The position where the zones start to overlap is at $x_{\max} = N x_1(\theta_0)$. Therefore,

$$x_{\max} = \frac{x_1(\theta_0)^2}{x_1\left(\theta_0 - \frac{\delta\theta_0}{2}\right) - x_1\left(\theta_0 + \frac{\delta\theta_0}{2}\right)}.$$

Since $\delta\theta_0 \ll \theta_0$, we can approximate

$$x_1\left(\theta_0 - \frac{\delta\theta_0}{2}\right) - x_1\left(\theta_0 + \frac{\delta\theta_0}{2}\right) \approx -\frac{\mathrm{d}x_1(\theta_0)}{\mathrm{d}\theta_0}\delta\theta_0 = \frac{Ab}{\sin^2(b\theta_0)}\delta\theta_0.$$

Combining the last two equations and substituting the $x_1(\theta_0)$ expression gives

$$x_{\max} = \frac{A\cos^2(b\theta_0)}{b\,\delta\theta_0} = \frac{2z_0\cos^2\theta_0}{\delta\theta_0}.$$