APhO 2024 – Theory Question No.1:

A1.

We have the equations of motion:

$$x(t) = v_0 \cos\theta\, t$$

$$y(t) = h + v_0 \sin\theta\, t - \frac{1}{2} g t^2$$

A2.

From the two equations above, we have:

$$t = \frac{x}{v_0 \cos\theta}$$

$$y = h + x\tan\theta - \frac{gx^2}{2v_0^2 \cos^2\theta}$$

$$y = h + x\tan\theta - \frac{gx^2}{2v_0^2}(1 + \tan^2\theta)$$

A3.

We have:

$$0 = \frac{gx^2}{2v_0^2}\tan^2\theta - x\tan\theta + y + \frac{gx^2}{2v_0^2} - h$$

The condition for this equation to have a real solution:

$$\Delta = x^2 - 4.\frac{gx^2}{2v_0^2}.\left(\frac{gx^2}{2v_0^2} + y - h\right) \geq 0$$

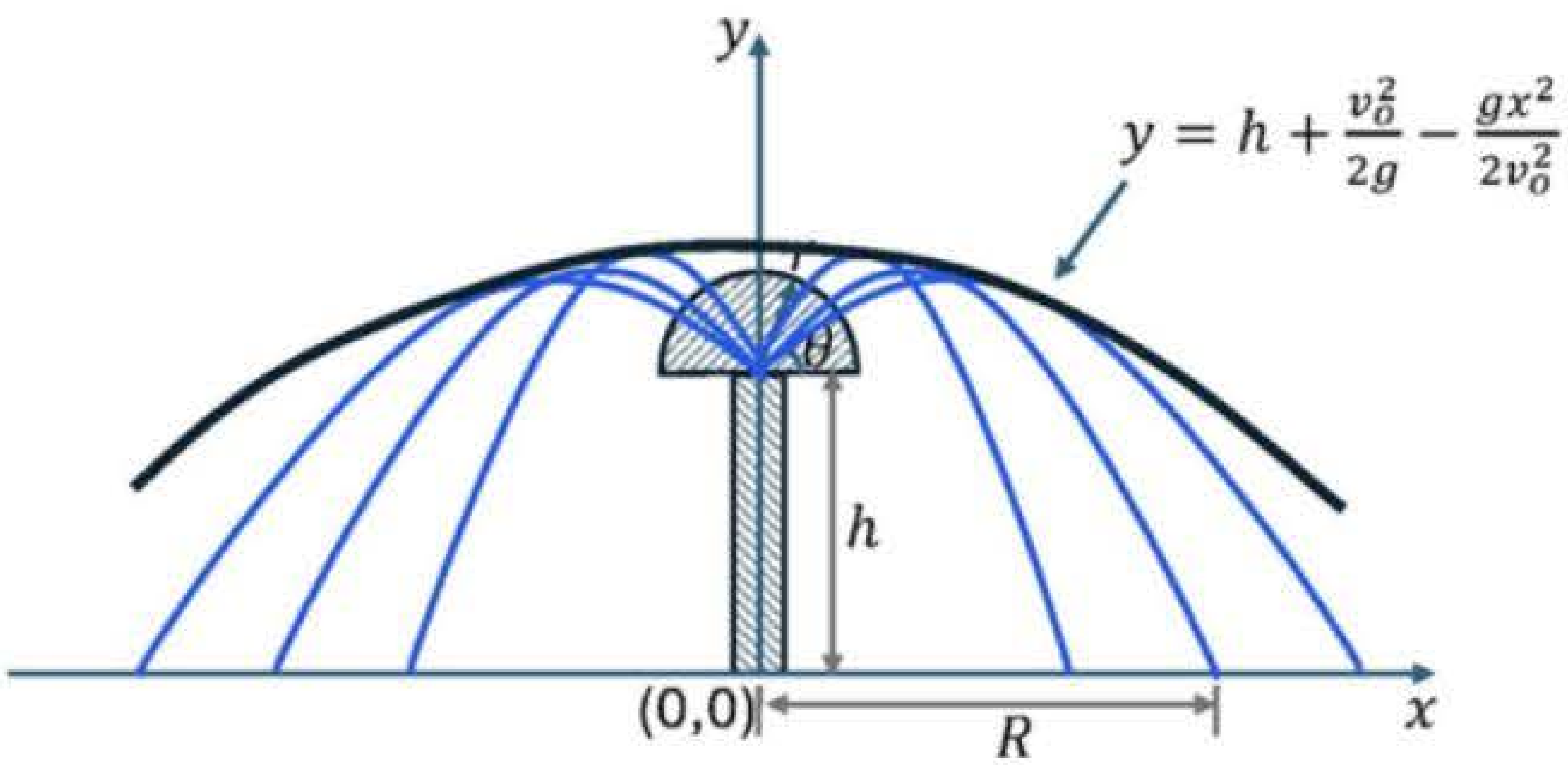
From here we clearly have:

$$y \leq h + \frac{v_0^2}{2g} - \frac{gx^2}{2v_0^2}$$

This means that the envelope over the water trajectories is:

$$y = h + \frac{v_0^2}{2g} - \frac{gx^2}{2v_0^2}$$

The figure of the envelope over the water trajectories:



A4.

The range of the water fountain can be calculated by applying the condition that $y(x, \theta) = 0$. Then, $R(\theta)$ would equal to x.

We have:

$$R(\theta) = \frac{\tan\theta + \sqrt{\tan^2\theta - 4h\frac{g(1+\tan^2\theta)}{2v_0^2}}}{\frac{g(1+\tan^2\theta)}{v_0^2}}$$

A5.

Substituting $h = 0$, we have:

$$R(\theta) = \frac{v_0^2}{g}\frac{\tan\theta + \sqrt{\tan^2\theta}}{1+\tan^2\theta} = \frac{v_0^2 \sin 2\theta}{g}$$

B1.

The elemental area of an annulus of radius R and width dR:

$$dA_w = 2\pi R(\theta) dR(\theta) = 2\pi R(\theta)|R'(\theta)|d\theta$$

B2.

The elemental area of water spray through the hemisphere from with area dA_w:

$$dA_H = 2\pi\rho(\theta)r^2\cos\theta\, d\theta$$

In order for the spray coverage pattern is uniform, $dA_H = dA_w$ and because $\theta < \frac{\pi}{4}$, the water rays do not intersect with each other.

$$2\pi\rho(\theta)r^2 \cos\theta \, d\theta = 2\pi R(\theta)R'(\theta)d\theta$$

We remove the absolute value sign because $R'(\theta) > 0$ in this case.

From there, we have:

$$\rho(\theta) = \frac{R(\theta)R'(\theta)}{r^2 \cos\theta}$$

B3.

We have from A5, using the condition $h = 0$:

$$R'(\theta) = \frac{dR}{d\theta} = \frac{2v_0^2 \cos 2\theta}{g}$$

This means that the fountain's holes distribution is:

$$\rho(\theta) = \frac{v_0^4}{g^2r^2}\frac{\sin 4\theta}{\cos\theta}$$