General Instructions for Theoretical Examination

Before the Exam

1. The allocated time for theoretical examination is 5 hours for answering 3 questions that carry 10 points each.
2. Please ensure that your student code matches the one on your desk. Place your ID / tag on the top right corner of the desk.
3. Kindly ensure that you have the following items on the desk:
   (i) A sealed envelope containing the Question Papers and Answer Scripts.
   (ii) A labelled envelope.
   (iii) Pens/relevant stationery (You need to bring in your own calculator).
   If you find that the items are incomplete, kindly raise your hands.
4. You must not open the sealed envelopes before the start of the examination.

During the Exam

1. You must use the dedicated answer scripts provided for writing your answers. Write your answers only in boxed answer scripts. If you need more boxed answer scripts, please raise your hand and ask an invigilator. Clearly write your student code and write which question you are attempting on each of your extra requested answer scripts.
2. **There are additional blank draft papers. These will not be marked.**
3. Try to be as concise as possible in your answers: use equations, logical operators, and sketches to illustrate your thoughts whenever possible. Avoid the use of long sentences.
4. Use an appropriate number of significant figures when stating numbers.
5. You are not allowed to leave your working place without permission. If you need any assistance (malfunctioning calculator, need to visit a restroom, insufficient Draft Papers, etc), please draw the attention of the invigilator by raising your hand.

At the end of the Exam

1. At the end of the examination, you must stop writing immediately.
2. Put all the Answer Scripts into the labelled envelope and seal it, including any blank Answer Scripts. It is your responsibility to ensure all Answer Scripts are inside the envelope and submitted to the invigilator.
3. Put question papers and draft papers into another envelope. You are not allowed to take any sheets of paper out of the examination hall.
4. Wait at your table in silence till your envelopes are collected. Once all envelopes are collected, your guide will escort you out of the examination area.

You must use the dedicated Answer Scripts provided for writing your answers. Write your answers only in boxed answer scripts. If you need more boxed answer scripts, please raise your hand and ask an invigilator. Clearly **write your student code** and **write which question number you are attempting** on each of your extra requested answer scripts.

There are additional blank draft papers. These will not be marked.

Physical Constants

Acceleration due to gravity	$g = 9.81\ \mathrm{m\,s^{-2}}$
Boltzmann constant	$k_B = 1.38 \times 10^{-23}\ \mathrm{J\,K^{-1}}$
Current Mass of the Sun	$M_S = 2.00 \times 10^{30}\ \mathrm{kg}$
Current Radius of the Sun	$R_S = 7.00 \times 10^{8}\ m$
Average Mass of a Galaxy	$M_{galaxy} = 1.5 \times 10^{12}\ M_S$
Magnitude of the electron charge	$e = 1.60 \times 10^{-19}\ \mathrm{C}$
Mass of the electron	$m_e = 9.11 \times 10^{-31}\ \mathrm{kg}$
Mass of the proton	$m_p = 1.67 \times 10^{-27}\ \mathrm{kg}$
Mass of the neutron	$m_n = 1.67 \times 10^{-27}\ \mathrm{kg}$
Permeability of free space	$\mu_0 = 4\pi \times 10^{-7}\ \mathrm{T\,m\,A^{-1}}$
Permittivity of free space	$\varepsilon_0 = 8.85 \times 10^{-12}\ \mathrm{F\,m^{-1}}$
Intrinsic impedance of free space	$Z_0 = 120\pi\ \Omega$
Planck's constant	$h = 6.63 \times 10^{-34}\ \mathrm{J\,s}$
Speed of light in vacuum	$c = 3.00 \times 10^{8}\ \mathrm{m\,s^{-1}}$
Stefan-Boltzmann constant	$\sigma = 5.67 \times 10^{-8}\ \mathrm{W\,m^{-2}\,K^{-4}}$
Universal Gravitational constant	$G = 6.67 \times 10^{-11}\ \mathrm{N\,m^{2}\,kg^{-2}}$

Geometry of Water Fountain

Introduction

Water sprayers are found at various places such as agriculture farms, green parks or urban areas for functional purposes or for aesthetic art installation. Consider a hemispherical shaped fountain sprayer of radius r at the height h from the ground as shown in a cross-sectional diagram in **Fig. 1**. Let the radius of the hemisphere to be small compared to the range R, hence can be treated as a point source. However there are $\rho(\theta)$ number of holes per unit area at angle θ. The water spurts in all directions at the same initial velocity v_o.

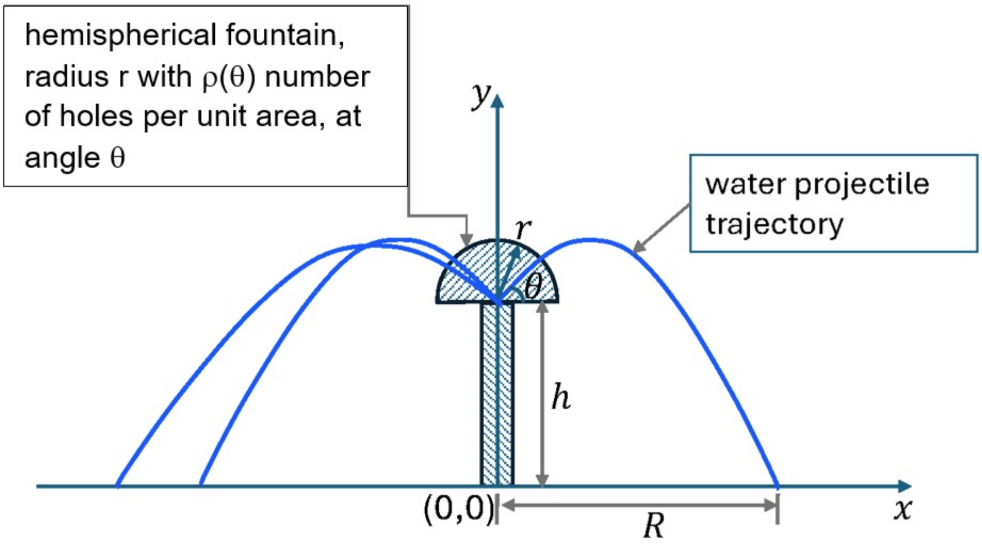

Fig. 1: Schematic cross section of a hemispherical shaped fountain sprayer.

Part A: Uniformly Distributed Holes on the Surface of the Hemisphere (6.0 points)

First consider the holes distribution on the surface of the hemisphere to be uniform (i.e. $\rho(\theta)$ is a constant).

A.1	Express the instantaneous position $x(t)$ and $y(t)$ of a water element (can be treated as a stream of particles) launched at velocity v_o at angle θ.	**0.5pt**

A.2	Derive the water trajectory relation $y = y(x, \theta)$ and express in a same type of trigonometric function (e.g. in terms of sine, cosine or tangent ONLY). Note that $\theta > 0$.	**1.0pt**

A.3	The envelope is the minimal boundary which encloses all water trajectories (i.e. no water will be found beyond this boundary). Derive the equation for the envelope for this system. Express your answer as y(x). Sketch the envelope over the trajectories.	**3.0pt**

A.4	Calculate the range $R = R(\theta)$ as a function of angle θ.	**1.0pt**

A.5	Letting $h \to 0$ in the range obtained in **A.4**, show that $R = \frac{v_o^2}{g}\sin\ 2\theta$.	**0.5pt**

Part B: Non-Uniformly Distributed Holes on the Surface of the Hemisphere (4.0 points)

Now, consider the area of holes per unit area $\rho(\theta)$ to be non-uniformly distributed and depended on the angle θ.

B.1	Using the results from part **A.4** above, calculate the elemental area dA_W of an annulus (ring) of radius R and width dR of the water hitting the ground. Express your answer in R, R' and dθ. [hint: consider R'=$\frac{dR}{d\theta}$]	**1.0pt**

B.2	Determine the relation for $\rho(\theta)$ that gives a uniform spray coverage pattern on the ground by considering elemental area on the hemisphere in proportional to the elemental area of the water spray on the ground. Express your answer in terms of R and its derivative. [hint: consider condition where $R\prime(\theta) > 0$].	**1.0pt**

B.3	Letting $h = 0$, calculate explicitly the relation for $\rho(\theta)$ that gives a uniform spray coverage pattern for $\theta < 45^\circ$.	**2.0pt**