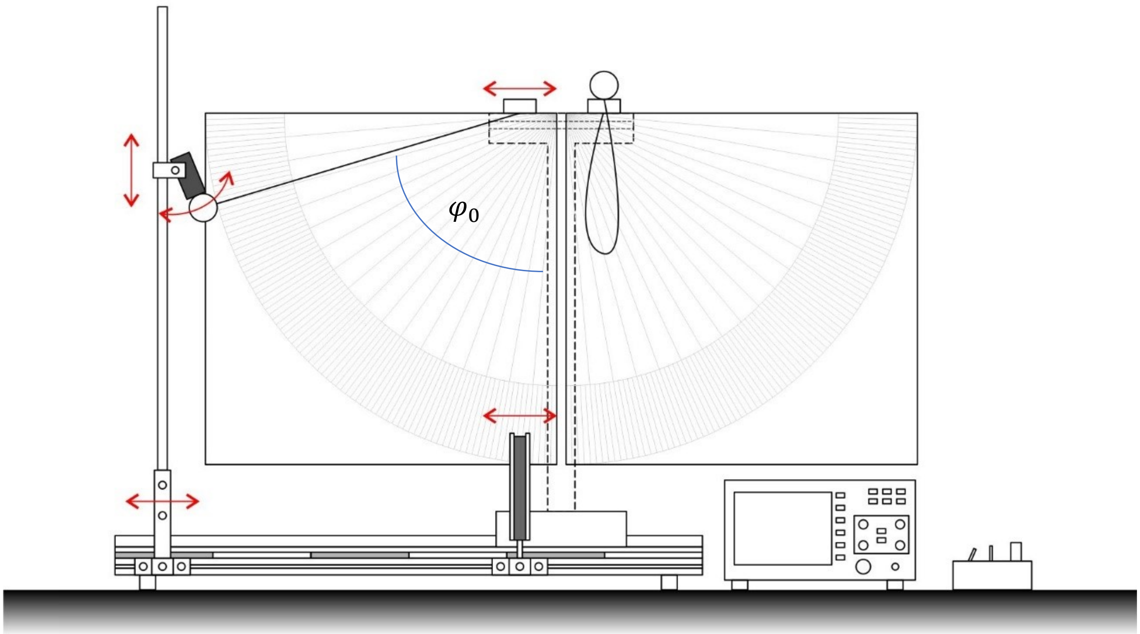
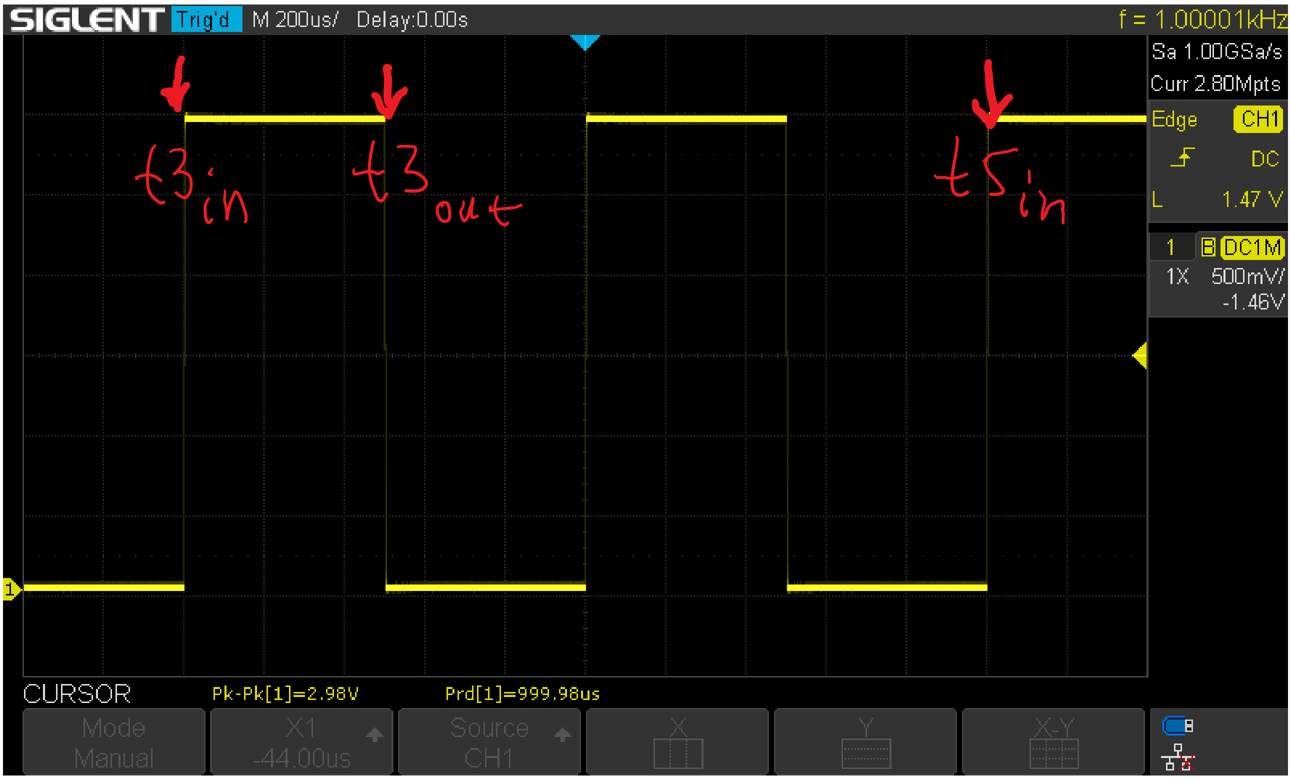
Q1 HERTZIAN CONTACT STRESS

PART A. LARGE ANGLE PENDULUM



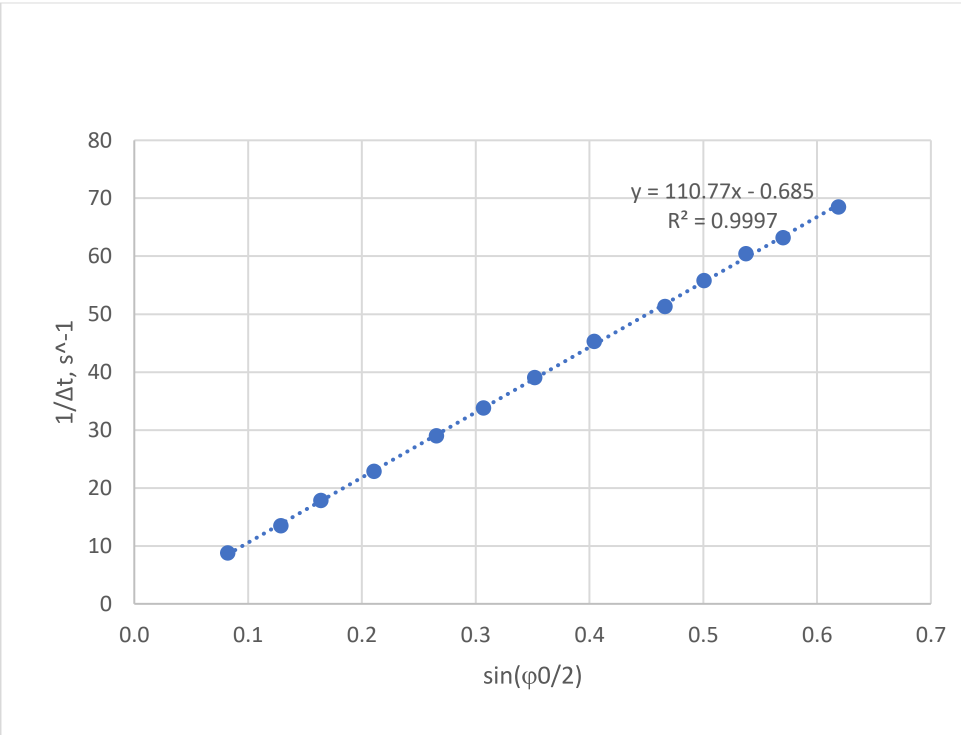
A.1. Δt ***and*** v



This record table includes both measurements A1 (Δt- time required for the fall to pass through the photogate) and B1($T - period\ of\ oscillation$) for each angle φ_0, in a single table, students can save time by doing both measurements at same angle.

Table №1

№	$\varphi_0, °$	$\sin\frac{\varphi_0}{2}$	$t3_{in}, s$	$t3_{out}, s$	$t5_{in}, s$	$\Delta t, s$	$\frac{1}{\Delta t}, s^{-1}$	T, s $t5_{in} - t3_{in}$	$v, m/s$
1	77.0	0.6188	1.257570	1.272170	2.51094	0.01460	68.49	1.25337	2.174
2	70.0	0.5701	1.227325	1.243150	2.45284	0.01583	63.19	1.22552	2.006
3	65.5	0.5376	1.218315	1.234875	2.43444	0.01656	60.39	1.21612	1.917
4	60.5	0.5006	1.203485	1.221420	2.40432	0.01794	55.76	1.20083	1.770
5	56.0	0.4664	1.188010	1.207500	2.37514	0.01949	51.31	1.18713	1.629
6	48.0	0.4041	1.170770	1.192860	2.34031	0.02209	45.27	1.16954	1.437
7	41.5	0.3519	1.158360	1.183990	2.31476	0.02563	39.02	1.15640	1.238
8	36.0	0.3069	1.147050	1.176660	2.29352	0.02961	33.77	1.14647	1.072
9	31.0	0.2654	1.140020	1.174520	2.27940	0.03450	28.99	1.13938	0.9200
10	24.5	0.2107	1.135300	1.179100	2.26742	0.04380	22.83	1.13212	0.7246
11	19.0	0.1639	1.115780	1.171840	2.24334	0.05606	17.84	1.12756	0.5662
12	14.9	0.1288	1.100050	1.174300	2.22485	0.07425	13.47	1.12480	0.4275
13	9.5	0.0822	1.107750	1.221600	2.23065	0.11385	8.78	1.12290	0.2789



graph №1: $\frac{1}{\Delta t}$ vs $\sin\frac{\varphi_0}{2}$

$$\text{grad}_{1/\Delta t} = 110.77\ s^{-1} \quad (1)$$

A.2. Speed of ball: $v = k_v \sin\frac{\varphi_0}{2}$

$$\frac{mv^2}{2} = mgl(1-\cos\varphi_0) = 2mgl\sin^2\frac{\varphi_0}{2}$$

$$v = 2\sqrt{gl}\sin\frac{\varphi_0}{2} \quad (2)$$

$$v = \frac{d}{\Delta t}$$

$$\frac{1}{\Delta t} = 2\frac{\sqrt{gl}}{d}\sin\frac{\varphi_0}{2} \quad (3)$$

$$\text{grad}_{1/\Delta t} = 2\frac{\sqrt{gl}}{d} = \frac{gT_0}{\pi d} = 110.77\ \text{s}^{-1} \quad (4)$$

$$d = 3.175 * 10^{-2}\ \text{m}$$

$$v = k_v \sin\frac{\varphi_0}{2} \quad (5)$$

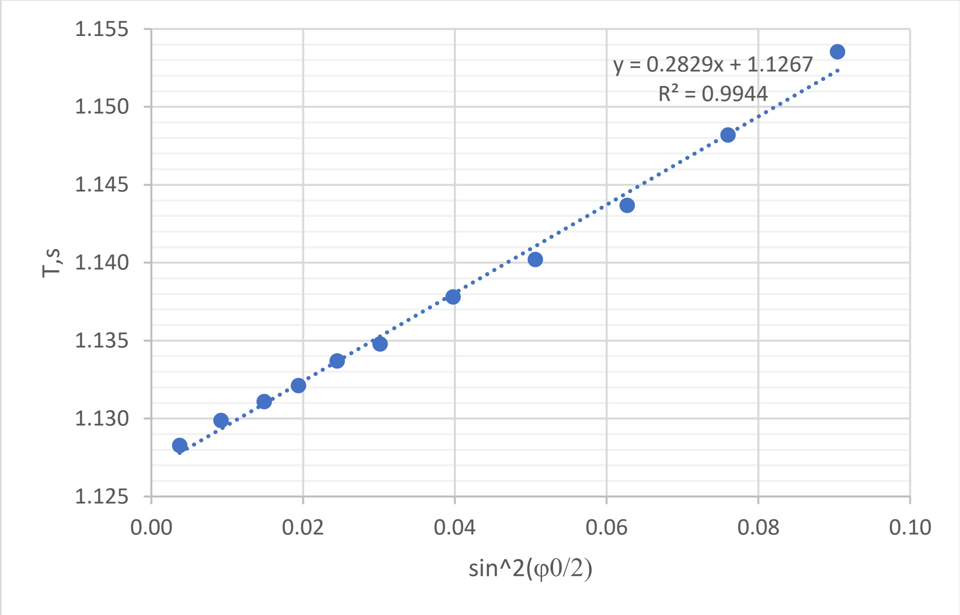
$$k_v = 3.517\frac{\text{m}}{\text{s}}$$

PART B. PERIOD OF THE OSCILLATION

B.1. PERIOD OF OSCILLATION: T_0, α, β

Table 2. Measurement at smaller angles

№	$\varphi_0, ^\circ$	$\sin\frac{\varphi_0}{2}$	$\sin^2\frac{\varphi_0}{2}$	$t3_{in}$, s	$t3_{out}$, s	$t5_{in}$, s	T, s	Δt, s	$\frac{1}{\Delta t}$, s^{-1}
1	35.0	0.3007	0.09042	1.153908	1.183840	2.307440	1.1535	0.02993	33.41
2	32.0	0.2756	0.07598	1.148388	1.181756	2.296584	1.1482	0.03337	29.97
3	29.0	0.2504	0.06269	1.138780	1.176372	2.282468	1.1437	0.03759	26.60
4	26.0	0.2250	0.05060	1.140252	1.182736	2.280460	1.1402	0.04248	23.54
5	23.0	0.1994	0.03975	1.133444	1.181036	2.271252	1.1378	0.04759	21.01
6	20.0	0.1736	0.03015	1.117848	1.175064	2.252636	1.1348	0.05722	17.48
7	18.0	0.1564	0.02447	1.133656	1.197872	2.267360	1.1337	0.06422	15.57
8	16.0	0.1392	0.01937	1.128688	1.204768	2.260812	1.1321	0.07608	13.14
9	14.0	0.1219	0.01485	1.131244	1.232416	2.262348	1.1311	0.10117	9.884
10	11.0	0.0958	0.00919	1.130564	1.269412	2.260456	1.1299	0.13885	7.202
11	7.0	0.0610	0.00373	1.127648	1.338152	2.255932	1.1283	0.21050	4.751



graph №2: T vs $\sin^2\frac{\varphi_0}{2}$

$$\sin^4\frac{30°}{2} = 4.5 * 10^{-3} \ll 1$$

$$T \approx T_0\left(1 + \alpha * \sin^2\frac{\varphi_0}{2}\right)$$

$$T_0 = 1.1267\ \mathrm{s}$$

$$\alpha T_0 = 0.2829\ \mathrm{s}$$

$$\alpha = \frac{0.2829}{1.1267} = 0.251$$

$$\alpha_{theor} = \frac{1}{4} = 0.250$$

$$\varepsilon_\alpha < 1\%$$

Experiment №1

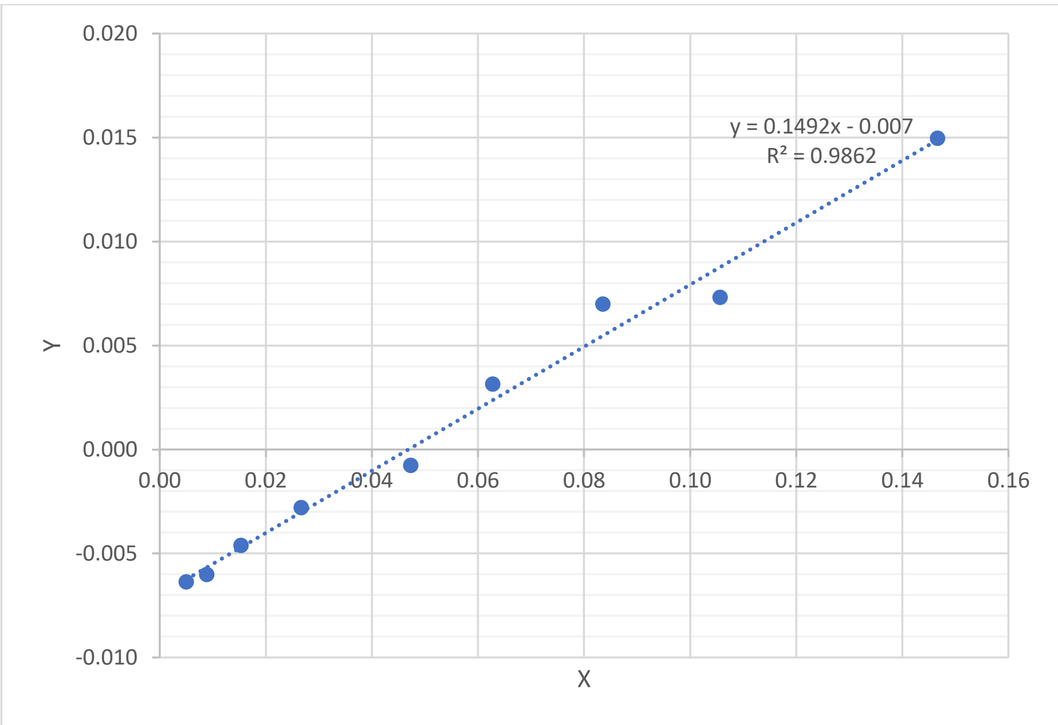
Table 3. Measurement at higher angles

№	φ_0, °	$t3_{in}$, s	$t5_{in}$, s	$\sin\frac{\varphi_0}{2}$	$\sin^2\frac{\varphi_0}{2}$	$X = \sin^4\frac{\varphi_0}{2}$	$T = t5_{in} - t3_{in}$, s	Y
1	77.0	1.257570	2.50900	0.6188	0.3829	0.1466	1.25143	0.0150
2	70.0	1.227325	2.45380	0.5701	0.3250	0.1056	1.22648	0.0073
3	65.5	1.218315	2.43430	0.5376	0.2890	0.08353	1.21599	0.0070
4	60.5	1.203485	2.40432	0.5006	0.2506	0.06279	1.20083	0.0031
5	56.0	1.188010	2.37514	0.4664	0.2176	0.04734	1.18713	-0.0008
6	48.0	1.170770	2.34031	0.4041	0.1633	0.02665	1.16954	-0.0028
7	41.5	1.158360	2.31476	0.3519	0.1238	0.01534	1.15640	-0.0046
8	36.0	1.147050	2.29352	0.3069	0.09420	0.00887	1.14647	-0.0060
9	31.0	1.140020	2.27940	0.2654	0.07044	0.00496	1.13938	-0.0064
10	24.5	1.135300	2.26742	0.2107	0.04440	0.00197	1.13212	-0.0063
11	19.0	1.115780	2.24334	0.1639	0.02686	0.000722	1.12756	-0.0060
12	14.9	1.100050	2.22485	0.1288	0.01658	0.000275	1.12480	-0.0058
13	9.5	1.107750	2.23065	0.0822	0.00676	0.000046	1.12290	-0.0051

$$T = T_0\left(1 + \alpha * \sin^2\frac{\varphi_0}{2} + \beta * \sin^4\frac{\varphi_0}{2}\right)$$

$$Y = \frac{T}{T_0} - 1 - \alpha * \sin^2\frac{\varphi_0}{2};\ X = \sin^4\frac{\varphi_0}{2}$$

$$Y = \beta X$$



graph №3: $Y\ vs\ X$

$$\beta = 0.149$$

$$\beta_{theor} = \frac{9}{64} = 0.141$$

$$\varepsilon_\beta = 6\%$$

B.2 Free fall acceleration in UB: g_{UB}

$$d = 31.75\ \mathrm{mm}$$

$$\mathrm{grad}_{1/\Delta t} = 2\frac{\sqrt{gl}}{d} = \frac{gT_0}{\pi d} = 110.77\ \mathrm{s}^{-1} \quad \text{from A.1 (4)}$$

$$T_0 = 1.1267\ \mathrm{s}$$

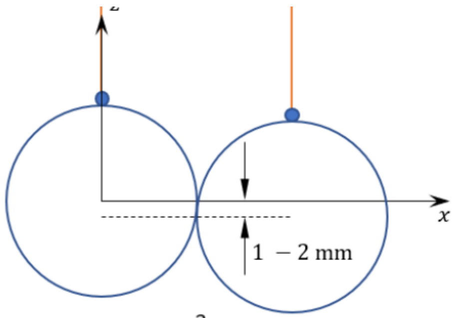
$$g_{exp} = \frac{\pi d * \mathrm{grad}_{1/\Delta t}}{T_0} = \frac{3.142 * 3.175 * 10^{-2} * 110.77}{1.1267} = 9.808\ \frac{\mathrm{m}}{\mathrm{s}^2}$$

$$g_{UB} = 9.804\ \frac{\mathrm{m}}{\mathrm{s}^2}$$

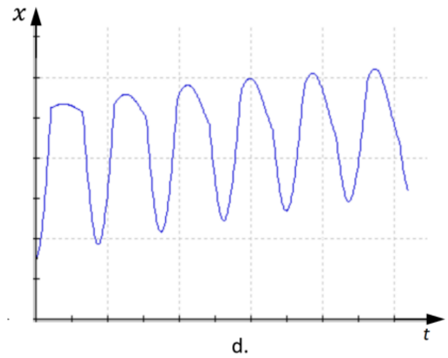
$$\varepsilon_g < 1\%$$

PART C: BEHAVIORS OF THE COLLISIONS

C1

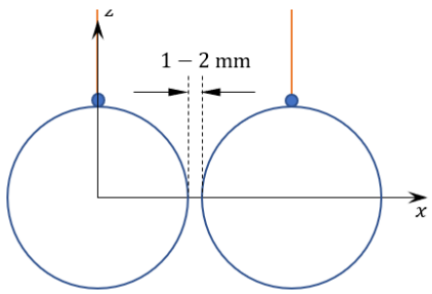


a.

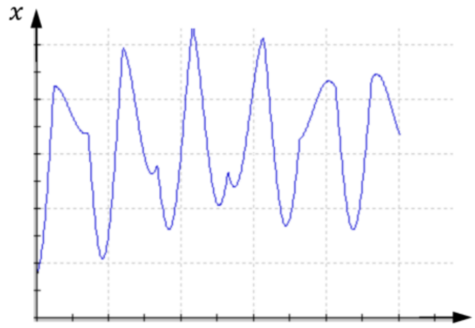


d.

C2

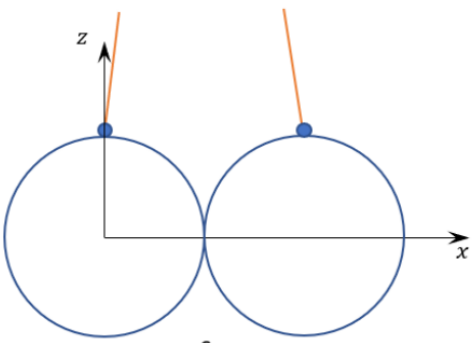


b.

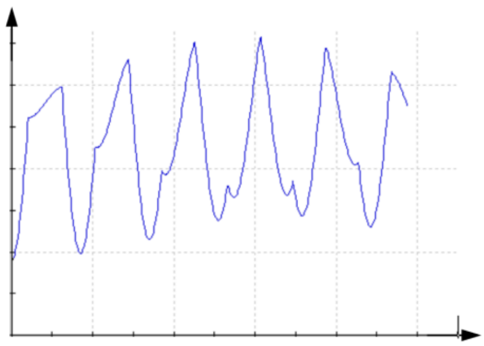


c. t

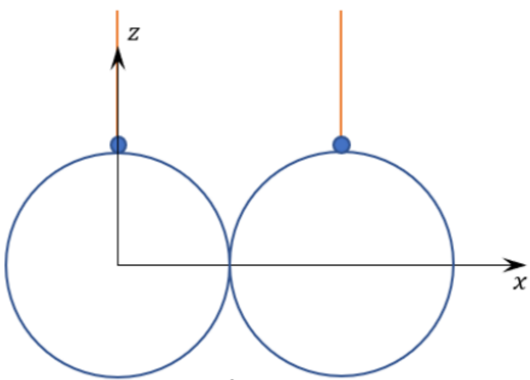
C3



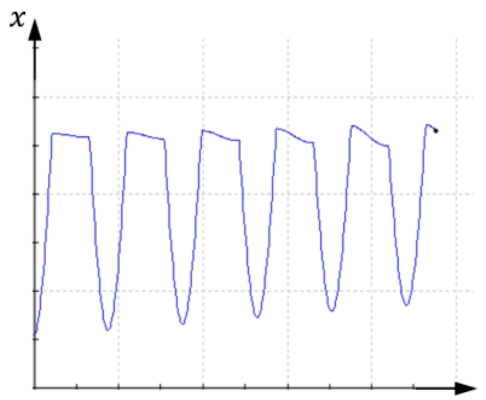
c.


e. t

C4



d.



a.

PART D: TIME OF COLLISIONS

D.1 Collision time equation: dimensional analysis

$$\tau = A * M^{\varepsilon_1} E^{\varepsilon_2} R^{\varepsilon_3} v^{\varepsilon_4} \quad (6)$$

$$M - \ mass; E - Young's\ modulus; R - radius; v - speed; A - dimensionless\ constant$$

$$\tau = M^{\varepsilon_1} M^{\varepsilon_2} L^{-\varepsilon_2} T^{-2\varepsilon_2} L^{\varepsilon_3} L^{\varepsilon_4} T^{-\varepsilon_4}$$

$$\begin{cases} M: 0 = \varepsilon_1 + \varepsilon_2 \\ L: \ 0 = -\varepsilon_2 + \varepsilon_3 + \varepsilon_4 \\ T: 1 = -2\varepsilon_2 - \varepsilon_4 \end{cases}$$

Three equations, Four exponents (ε_1, $\varepsilon_2, \varepsilon_3, \varepsilon_4$). You have to determine some exponent in an experimental way.

D.2 Exponent of speed

$$\tau = const * v^{\varepsilon_4}$$

For expression (5)

$$v = k_v \sin\frac{\varphi_0}{2}$$

$$k_v = 3.517\,\frac{\text{m}}{\text{s}}$$

$$m = 131.48\text{ g};\ \ d = 31.73\text{ mm}; l = 29.7\text{ cm}; \text{sound speed in steel } c = \sqrt{\frac{E'}{\rho}} = 5180\,\frac{\text{m}}{\text{s}}$$

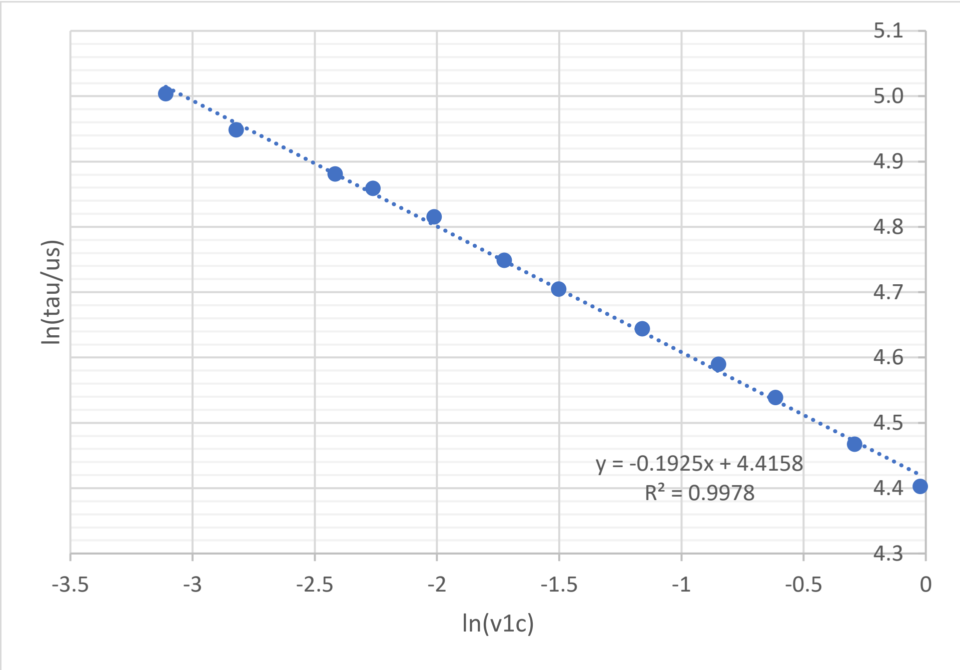
Table 4. Time of collisions as a function of speed

$\varphi_0, {}^\circ$	corrected	v, m/s	v_{1c}, m/s	$\tau, \mu s$	ln (v_{1c}, m/s)	Ln ($\tau, \mu s$)
70.0	69.98	1.96	0.978	81.7	-0.0227	4.403
52.0	51.99	1.49	0.747	87.14	-0.2916	4.468
37.0	36.99	1.08	0.541	93.58	-0.6148	4.539
29.1	29.09	0.86	0.428	98.47	-0.8483	4.590
21.2	21.19	0.63	0.313	104.00	-1.1600	4.644
15.0	15.00	0.44	0.222	110.50	-1.5030	4.705
12.0	12.00	0.36	0.178	115.50	-1.7252	4.749
9.0	9.00	0.27	0.134	123.40	-2.0121	4.815
7.0	7.00	0.21	0.104	128.90	-2.2630	4.859
6.0	6.00	0.18	0.089	131.80	-2.4169	4.881
4.0	4.00	0.12	0.059	141.00	-2.8222	4.949
3.0	3.00	0.09	0.045	149.00	-3.1098	5.004
1.0	1.0	0.03	0.015	207.00	-4.2083	5.333

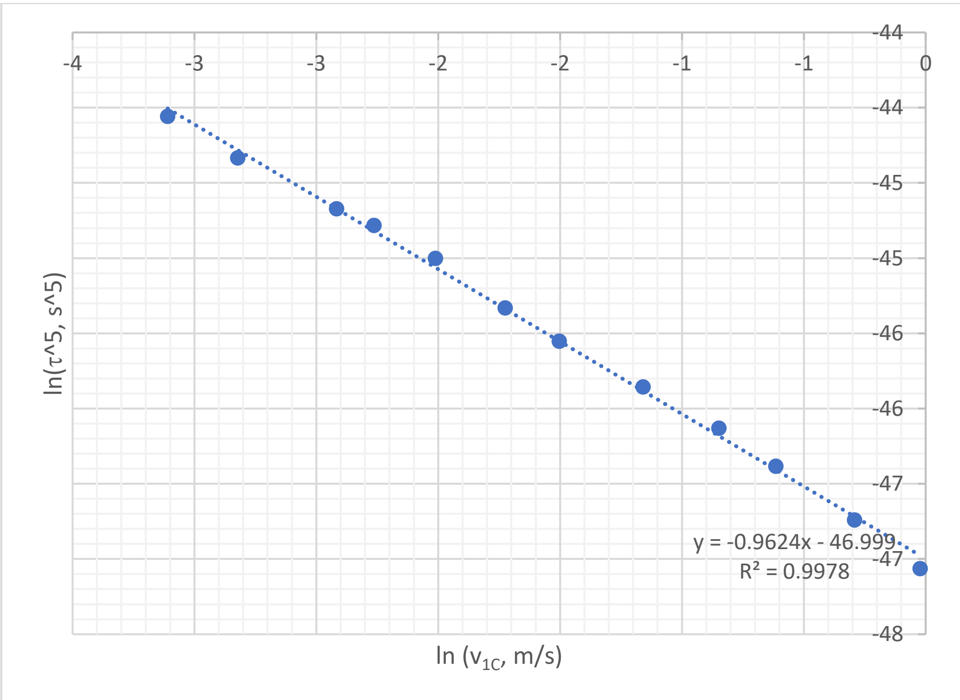
$$v_{1c} = \frac{v}{2}$$

$$\varepsilon_4 = -0.193\ (3.5\ \%)$$

$$\varepsilon_{4theor} = -0.200$$

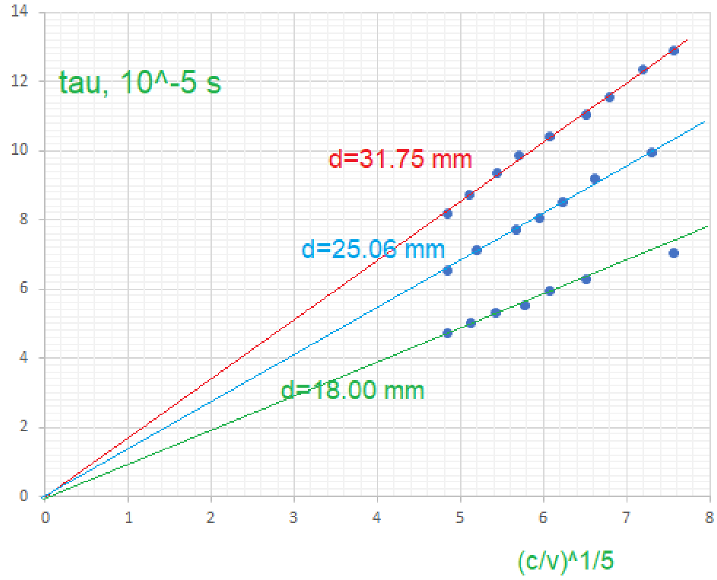


graph №4: $\tau\ vs\ v_{1c}$



$$\varepsilon_4 = -0.96/5 \approx -1/5\ (rel.error = 4\ \%)$$

Exponent of radius



graph №5: $\tau\ vs\ \left(\frac{c}{v}\right)^{\frac{1}{5}}$ at different diameters of balls

$$grad1: grad2: grad3 = 12: 9.5: 6.8 = 1.76: 1.40: 1$$

$$d_1: d_2: d_3 = 31.75: 25.06: 18.00 = 1.76: 1.40: 1$$

Result:

$$\tau \sim (R_1 + R_2)$$

D.3 Solution of exponents:

$$\varepsilon_1 = \frac{2}{5}$$

$$\varepsilon_2 = -\frac{2}{5}$$

$$\varepsilon_3 = -\frac{1}{5}$$

$$\varepsilon_4 = -\frac{1}{5}$$

$$\tau^5 \sim M^2 E^{-2} R^{-1} v^{-1}$$

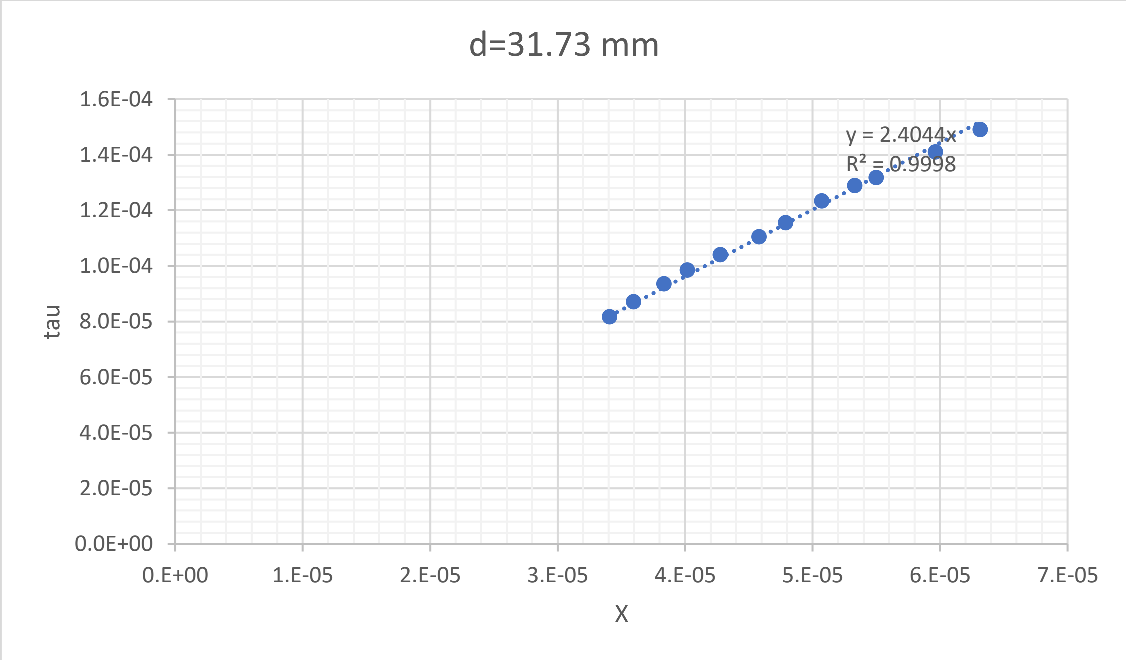
$$\tau^5 \sim \frac{M^2}{E^2 R v}$$

$$\tau^5 \sim \frac{\rho^2 R^6}{E^2 R v} \sim \frac{R^6}{c^4 R v} \sim \frac{R^5}{c^5} * \frac{c}{v}$$

$$\tau \sim \frac{R_1 + R_2}{c} * \left(\frac{c}{v}\right)^{\frac{1}{5}} = X$$

D.4 Numerical value of A in (6)

$$m = 131.48\text{ g};\ \ d = 31.73\text{ mm}; l = 29.7\text{ cm}; c = 5180\frac{\text{m}}{\text{s}}$$



graph №6: $\tau\ vs\ X$

$$A_{exp} = 2.40$$

$$\tau = 2.40\frac{R_1 + R_2}{c} * \left(\frac{c}{v}\right)^{\frac{1}{5}} \qquad (6')$$

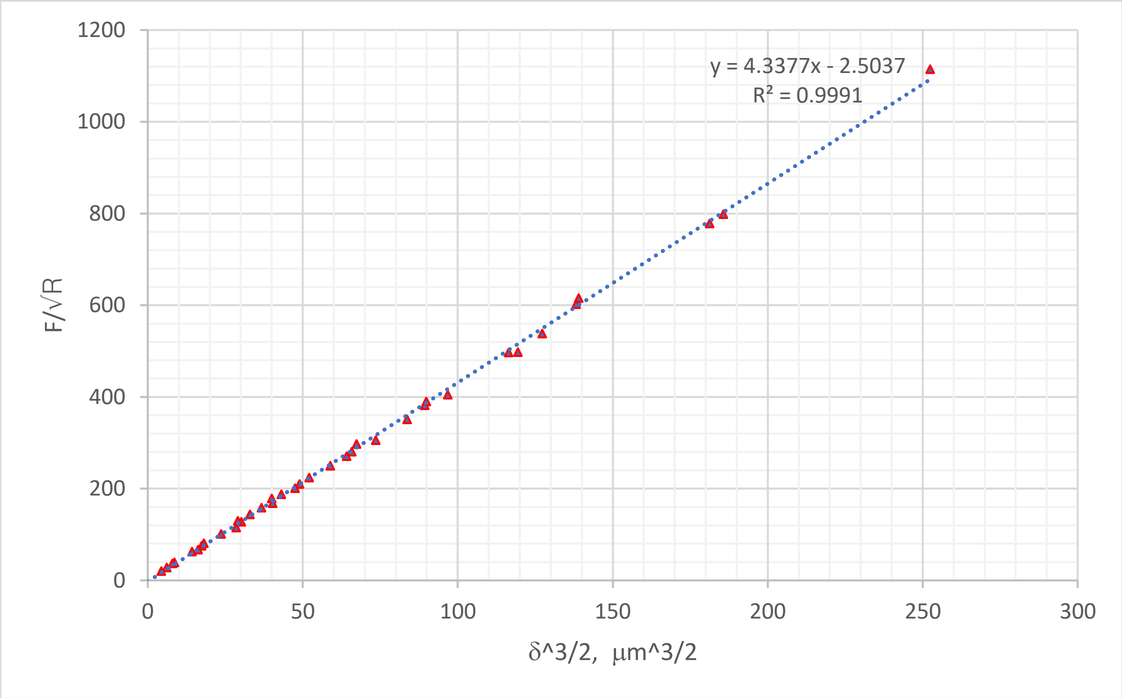
PART E: THE PARAMETERS OF HERTZ DEFORMATION

Table 5. Force, Hertz Deflection, Hertz Radius, Hertz Pressure.

Balls	φ_0, ,	v, m /s	τ, μs	Δp, Ns	F_{av}, N	δ, μm	$\delta^{\frac{1}{5}}$, μm$^{1.5}$	$\frac{F_{av}}{\sqrt{R}}$, Nm$^{-0.5}$	a, mm	P_0, GPa
d/mm	70.0	1.96	81.70	0.2565	3139	39.93	252.3	1114.2	0.55	4.88
31.74	52.0	1.49	87.14	0.1960	2249	32.55	185.7	798.4	0.50	4.37
m/g	37.0	1.08	93.58	0.1419	1516	25.30	127.3	538.1	0.43	3.83
131.48	29.1	0.86	98.47	0.1123	1141	21.08	96.8	404.9	0.40	3.48
	21.2	0.63	104.00	0.0822	791	16.30	65.8	280.7	0.35	3.08
	15.0	0.44	110.50	0.0584	528	12.29	43.1	187.5	0.31	2.70
	12.0	0.36	115.50	0.0467	405	10.29	33.0	143.6	0.28	2.47
	9.0	0.27	123.40	0.0351	284	8.25	23.7	100.9	0.25	2.19
	7.0	0.21	128.90	0.0273	212	6.71	17.4	75.2	0.23	1.99
	6.0	0.18	131.80	0.0234	178	5.88	14.3	63.0	0.21	1.87
	4.0	0.12	141.00	0.0156	111	4.19	8.6	39.3	0.18	1.60
	3.0	0.09	149.00	0.0117	79	3.32	6.1	27.9	0.16	1.43
d/mm	70.0	1.958	65.43	0.1274	1947	32.02	181.2	778.1	0.44	4.87
25.42	47.1	1.362	71.20	0.0886	1245	24.24	119.4	497.4	0.38	4.2
m/g	30.1	0.909	77.27	0.0591	765	17.55	73.5	305.8	0.32	3.54
67.55	23.5	0.693	80.40	0.0451	561	13.93	52.0	224.2	0.29	3.22
	18.5	0.552	85.20	0.0359	422	11.76	40.3	168.5	0.26	2.92
	13.8	0.407	91.85	0.0265	288	9.34	28.6	115.2	0.23	2.59
	8.4	0.257	99.70	0.0167	168	6.41	16.2	67.1	0.19	2.13
d/mm	70.0	2.022	47.15	0.0497	1054	23.84	116.4	496.8	0.32	4.91
18.00	50.5	1.523	50.25	0.0374	745	19.13	83.7	351.0	0.28	4.35
m/g	37.5	1.142	53.00	0.0281	529	15.13	58.9	249.6	0.25	3.89
24.57	27.5	0.849	55.10	0.0209	379	11.70	40.0	178.5	0.22	3.47
	21.1	0.653	59.25	0.0161	271	9.68	30.1	127.7	0.2	3.11
	15.0	0.439	62.80	0.0108	172	6.89	18.1	81.0	0.18	2.72
	7.0	0.226	70.20	0.0056	79	3.97	7.9	37.3	0.13	2.04
	4.0	0.136	78.80	0.0033	42	2.68	4.4	20.0	0.1	1.63
R1, R2/mm	70.0	1.956	54.89	0.0809	1475	26.83	139.0	615.2	0.39	4.71
9.00	69.0	1.931	55.37	0.0799	1443	26.73	138.2	602.3	0.38	4.68
15.88	46.5	1.346	59.61	0.0557	934	20.06	89.8	389.9	0.33	4.04
m1, m2/g	45.9	1.329	60.13	0.0550	915	19.98	89.3	381.8	0.33	4.02
24.57	36.5	1.068	62.00	0.0442	713	16.55	67.3	297.4	0.3	3.7
131.18	34.2	1.002	63.92	0.0415	649	16.02	64.1	270.8	0.29	3.58

Experiment №1 English (Official)

R, mm	27.4	0.807	66.30	0.0334	504	13.38	49.0	210.3	0.27	3.29
5.74	26.5	0.781	67.10	0.0323	482	13.11	47.5	201.1	0.27	3.24
μ, g	21.5	0.636	69.46	0.0263	379	11.04	36.7	158.1	0.25	2.99
20.69	18.0	0.533	70.90	0.0221	311	9.45	29.1	129.9	0.23	2.8



graph №7: *Unified graph for balls*

Equivalent mass

$$\mu = \frac{m_1 m_2}{m_1 + m_2} \qquad (7)$$

$$v_{relarive} = v_{1x} - v_{2x} = v_1 - 0 = v_1$$

$$\Delta p = 2\mu v_{\text{relative}} \qquad (8)$$

$$F_{av} = \frac{\Delta p}{\tau} \qquad (9)$$

$$K_{\text{system,C}} = \frac{1}{2}\mu v_{\text{relative}}^2 \quad (10)$$

$$K_{\text{system,C}} = F_{av}\delta$$

$$\delta = \frac{K_{\text{system,C}}}{F_{av}} \qquad (11)$$

Deflection

$$\delta \lesssim \frac{K_{\text{system,C}}}{F_{av}}$$

$$\delta \lesssim \frac{v_{\text{relative}}}{4}\tau \quad (12)$$

Hertz radius

From fig Q1:

$$a^2 = R^2 - \left(R - \frac{\delta}{2}\right)^2 \approx R\delta$$

$$a = \sqrt[3]{\frac{3FR}{4E'}} = \sqrt{R\delta} \quad \text{Hertz radius} \quad (13)$$

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} \qquad (14)$$

$$\delta = \sqrt[3]{\frac{9F_{av}^2}{16RE'^2}} \quad \text{Hertz deformation} \quad (15)$$

Reduced Youngs Modulus

$$\frac{1}{E'} = \frac{1-\nu_1^2}{E_1} + \frac{1-\nu_2^2}{E_2} = 2\frac{1-\nu_1^2}{E_1}$$

Poisson’s ratio

$$\nu_1 = \nu_2 = 0.3$$

Hertz pressure

$$P_{av} = \frac{1}{\pi a^2}\int_0^a P(r)2\pi r dr = -P_0\int_0^a \left(1 - \frac{r^2}{a^2}\right)^{\frac{1}{2}} d\left(1 - \frac{r^2}{a^2}\right) = \frac{2}{3}P_0$$

$$P_0 = \frac{3}{2}P_{av} \qquad (16)$$

$$P_0 = \frac{2}{\pi}E' * \left(\frac{\delta}{R}\right)^{\frac{1}{2}} \qquad (17)$$

$$F_{max} = \frac{3}{2}F_{av} \qquad (18)$$

$$F_{av} \sim \sqrt{R\delta^3} \qquad (19)$$