Theoretical Question 2

Motion of an Electric Dipole in a Magnetic Field

In the presence of a constant and uniform magnetic field $\vec{B}$, the translational motion of a system of electric charges is coupled to its rotational motion. As a result, the conservation laws for the *momentum* and the *component of the angular momentum* along the direction of $\vec{B}$ are modified from the usual form. This is illustrated in this problem by considering the motion of an ***electric dipole*** made of two particles of equal mass m and carrying *charges* q and $-q$ respectively ($q > 0$). The two particles are connected by a rigid insulating rod of *length* ℓ, the mass of which can be neglected. Let $\vec{r}_1$ be the *position vector* of the particle with charge q, $\vec{r}_2$ that of the other particle and $\vec{\ell} = \vec{r}_1 - \vec{r}_2$. Denote by $\vec{\omega}$ the *angular velocity* of the rotation around the center of mass of the dipole. Denote by $\vec{r}_{CM}$ and $\vec{v}_{CM}$ the *position* and the *velocity* vectors of the *center of mass* respectively. *Relativistic effects and effects of electromagnetic radiation can be neglected.*

Note that the *magnetic force* acting on a particle of charge q and velocity $\vec{v}$ is $q\vec{v}\times\vec{B}$, where the cross product of two vectors $\vec{A}_1\times\vec{A}_2$ is defined, in terms of the x, y, z, components of the vectors, by

$$(\vec{A}_1\times\vec{A}_2)_x = (\vec{A}_1)_y(\vec{A}_2)_z - (\vec{A}_1)_z(\vec{A}_2)_y$$
$$(\vec{A}_1\times\vec{A}_2)_y = (\vec{A}_1)_z(\vec{A}_2)_x - (\vec{A}_1)_x(\vec{A}_2)_z$$
$$(\vec{A}_1\times\vec{A}_2)_z = (\vec{A}_1)_x(\vec{A}_2)_y - (\vec{A}_1)_y(\vec{A}_2)_x.$$

(1) **Conservation Laws**

(a) Write down the *equations of motion* for the *center of mass* of the dipole and for the *rotation around* the center of mass by computing the total force and the total torque with respect to the center of mass acting on the dipole.

(b) From the equation of motion for the center of mass, obtain the ***modified** form* of the conservation law for the *total momentum*. Denote the corresponding modified conserved quantity by $\vec{P}$. Write down an *expression* in terms of $\vec{v}_{CM}$ and $\vec{\omega}$ for the conserved *energy E*.

(c) The angular momentum consists of two parts. One part is due to the motion of the center of mass and the other is due to rotation around the center of mass. From the modified form of the conservation law for the total momentum and the equation of motion of the rotation around the center of mass, prove that the quantity J as defined by

$$J = (\vec{r}_{CM}\times\vec{P} + I\vec{\omega})\cdot\hat{B}$$

is conserved.

Note that

$$\vec{A}_1 \times \vec{A}_2 = -\vec{A}_2 \times \vec{A}_1$$

$$\vec{A}_1 \cdot (\vec{A}_2 \times \vec{A}_3) = (\vec{A}_1 \times \vec{A}_2) \cdot \vec{A}_3$$

$$\vec{A}_1 \times (\vec{A}_2 \times \vec{A}_3) = (\vec{A}_1 \cdot \vec{A}_3)\vec{A}_2 - (\vec{A}_1 \cdot \vec{A}_2)\vec{A}_3$$

for any three vectors $\vec{A}_1$, $\vec{A}_2$ and $\vec{A}_3$. Repeated application of the above first two formulas may be useful in deriving the conservation law in question.

In the following, let $\vec{B}$ be in the z-direction.

(2) **Motion in a Plane Perpendicular to $\vec{B}$**

Suppose initially the center of mass of the *dipole* is *at rest* at the origin, $\vec{\ell}$ points in the x-direction and the *initial angular velocity* of the dipole is $\omega_0 \hat{z}$ ($\hat{z}$ is the unit vector in the z-direction).

(a) If the magnitude of ω_0 is smaller than a *critical value* ω_c, the dipole will not make *a full turn* with respect to its center of mass. Find ω_c.

(b) For a general $\omega_0 > 0$, what is the *maximum distance* d_m in the x-direction that the center of mass can reach?

(c) What is the *tension* on the rod? Express it as a function of the angular velocity ω.